

# Suppression of turbulent dynamo in time irreversible turbulence

A.V. Kopyev<sup>1</sup>, A.S. Il'yn<sup>1,2</sup>, V.A. Sirota<sup>1</sup> and K.P. Zybin<sup>1,2</sup>\*

<sup>1</sup> *P.N.Lebedev Physical Institute of RAS, 119991, Leninskij pr.53, Moscow, Russia*

<sup>2</sup> *National Research University Higher School of Economics, 101000, Myasnitskaya 20, Moscow, Russia*

The conventional theory of magnetic field generation in a turbulent flow considers time-reversible random flows. However, real turbulent flows are known to be time irreversible: the presence of energy cascade is an intrinsic property of turbulence. We generalize the 'standard' model to account for the irreversibility. We show that even small time asymmetry leads to significant suppression of the dynamo effect at low magnetic Prandtl numbers, increases the generation threshold and may even make generation impossible for any magnetic Reynolds number. We calculate the magnetic energy increment as a function of the parameters of the flow.

*Introduction.* The theory of passive magnetic field transport in a turbulent flow has numerous applications and, in particular, is likely to explain the existence of magnetic field in many astrophysical objects [1–5]. The basic idea of turbulent dynamo is that the magnetic lines stretch as they are carried by turbulent motion. However, a very wide range of magnetized astrophysical objects of different scales implies very different parameters of the medium. A small-scale turbulent dynamo, i.e., a dynamo that takes place at scales much less than the integral scale of turbulence [6, 7], depends on two basic parameters: the hydrodynamic Reynolds number  $Re$  and the magnetic Reynolds number  $Rm$ . Their ratio  $Rm/Re=Pm$  is called magnetic Prandtl number: it indicates whether the viscous scale  $r_\nu$  of turbulence is larger ( $Pm > 1$ ) or smaller than the resistive scale  $r_d$ . While both  $Rm$  and  $Re$  are large in cosmic plasmas, their ratio varies crucially, and  $Pm$  is either very large or very small in astrophysical objects. The Prandtl number determines the small-scale dynamo mechanism [8].

The high- $Pm$  turbulence is observed in interstellar and intergalactic media [8, 9]. In these objects, the characteristic scale of magnetic field generation is deep inside the viscous range of turbulence, so the stretching of magnetic lines is exponential [10]; the existence of the dynamo effect for this case is confirmed by different theoretical approaches [6, 11, 12] and by numerical simulations [13].

To the contrary, in stellar and planetary magnetism, in protostellar disks etc., one observes low- $Pm$  media ( $Pm \sim 10^{-3} \div 10^{-7}$ ) [8, 14]. In this case, small-scale dynamo can occur only in the inertial range of turbulence, at scales  $\rho \gtrsim r_d \gg r_\nu$ . The existence of this effect is still questionable. Direct numerical simulations are very difficult to perform in this range of parameters [15, 16]; some authors report the existence of generation, and even the existence of a universal limit  $Rm_c$  that guarantees the dynamo for any  $Re$  [17, 18], although their estimates for the threshold  $Rm_c$  differ essentially, and the values of  $Pm$  and especially  $Re$  that they achieve ( $Pm \gtrsim 10^{-3}$ ,  $Re \lesssim 10^5$ ) are still far from the observed in astrophysic objects. Simulations performed by means of the shell model [19–21] and the implicit large-eddy simulations [14, 22–24] allow to get small-scale dynamo at more realistic parameters, however, the results may depend significantly on the model assumptions [25, 26]. Observations of the Sun show the presence of small-scale magnetic field, but it is unclear whether it is, indeed, generated in the inertial range by turbulent motion [27], or it is merely a result of fragmentation of the large-scale field produced by some other process; Stenflo [28, 29] argues that the latter is more likely. Theoretical consideration predicts the possibility of the dynamo effect in the low- $Pm$  limit [30]. Kazantsev [12] has found the dependence of the growth increment on the scaling properties of isotropic incompressible flow with infinite  $Re$  (infinite integral scale of turbulence), under the assumption of Gaussian and delta correlated velocity statistics. In later papers, the model was extended for the case of finite  $Re$  [31–38].

However, these theoretical works consider only time-reversible random flows. Actually, the time asymmetry is associated with the third-order velocity correlator, which is zero in the classical Kazantsev model. Meanwhile, the real turbulent flows are known to be time irreversible: this follows from the existence of energy cascade [39, 40]. The account of small time irreversibility in the high- $Pm$  limit has shown the decrease of all magnetic field statistical moments growth rate [4, 42, 43]. So, there is a reasonable question: how does it affect the magnetic field generation in the low- $Pm$  case?

In this Letter, we consider the influence of the third order velocity correlator on the ability of a low- $Pm$  conductive fluid to generate magnetic field. We find the instability criterion and show that the magnetic generation in a turbulent flow may be suppressed completely if the third correlator is large enough. We show that in this case, even for infinite  $Rm$ , there is no generation for  $Pm$  small enough but finite.

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\* Electronic addresses: kopyev@lpi.ru, asil72@mail.ru, sirota@lpi.ru, zybin@lpi.ru

*Definitions and the  $V^3$  model.* The evolution of magnetic field in a (random) velocity field  $\mathbf{v}(\mathbf{r}, t)$  is described by the equation

$$\frac{\partial \mathbf{B}(\mathbf{r}, t)}{\partial t} = \text{rot}[\mathbf{v}(\mathbf{r}, t) \times \mathbf{B}(\mathbf{r}, t)] + \kappa \Delta \mathbf{B}(\mathbf{r}, t) \quad (1)$$

where  $\kappa$  is the magnetic diffusivity. The magnetic field energy in the problems under consideration is smaller than the kinetic energy of the flow at all scales, and the feedback influence of the magnetic field on the velocity dynamics is proportional to  $B^2$ , so one can neglect this feedback and consider  $\mathbf{v}(\mathbf{r}, t)$  as stationary random process with given statistics.

We consider isotropic and homogenous stationary flow in incompressible fluid. The Kazantsev equation for the second-order magnetic field correlator was derived for Gaussian  $\delta$ -correlated in time velocity field,

$$\langle v_i(\mathbf{r}, t) v_j(\mathbf{r}_1, t_1) \rangle = D_{ij}(\mathbf{r} - \mathbf{r}_1) \delta_\epsilon(t - t_1) \quad (2)$$

Here  $\delta_\epsilon$  is not a 'physical' singularity but a regularized  $\delta$  function: a narrow peak with unitary square and width smaller than all physical scales. To avoid ambiguities, for any real (short-correlated) velocity field one can define  $D_{ij}$  by the integral

$$D_{ij}(\boldsymbol{\rho}) = \int \langle v_i(\mathbf{r}, t) v_j(\mathbf{r} + \boldsymbol{\rho}, t + \tau) \rangle d\tau \quad (3)$$

(It does not depend on  $\mathbf{r}$  and  $t$  because of space and time homogeneity.) To take account of the third-order velocity correlator, we introduce

$$F_{ijk}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) = \int \langle v_i(\mathbf{r}, t) v_j(\mathbf{r} + \boldsymbol{\rho}_1, t_1) v_k(\mathbf{r} + \boldsymbol{\rho}_2, t_2) \rangle dt_1 dt_2 \quad (4)$$

Now we use the  $V^3$  model [4, 44, 45]. It is a generalization of the Kazantsev-Kraichnan model [12, 46] and implies two assumptions: 1) the third order correlator is assumed to be singular in time:

$$\langle v_i(\mathbf{r}, t) v_j(\mathbf{r} + \boldsymbol{\rho}_1, t_1) v_k(\mathbf{r} + \boldsymbol{\rho}_2, t_2) \rangle = F_{ijk}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) \frac{\delta_\epsilon(t-t_1) \delta_\epsilon(t-t_2) + \delta_\epsilon(t-t_1) \delta_\epsilon(t_1-t_2) + \delta_\epsilon(t-t_2) \delta_\epsilon(t_1-t_2)}{3}, \quad (5)$$

and 2) the higher order correlators are set to zero. We will discuss the validity of these assumptions a few strings later.

Under these assumptions, we derive the generalized Kazantsev equation (see Supplementary for derivation):

$$\begin{aligned} \frac{\partial}{\partial t} G &= 2\kappa \left( G''_{\rho\rho} + \frac{4G'_\rho}{\rho} \right) + bG''_{\rho\rho} + \left( b' + 4\frac{b}{\rho} \right) G'_\rho + \left( b'' + 4\frac{b'}{\rho} \right) G \\ &+ 2cG'''_{\rho\rho\rho} + 3 \left( c' + \frac{4c}{\rho} \right) G''_{\rho\rho} + \left( 3c'' + \frac{12c'}{\rho} + \frac{8c}{\rho^2} + \frac{8d}{\rho} \right) G'_\rho + \left( c''' + \frac{6c''}{\rho} + \frac{4c'}{\rho^2} - \frac{4c}{\rho^3} + \frac{4d'}{\rho} + \frac{12d}{\rho^2} \right) G \end{aligned} \quad (6)$$

Here

$$G(\boldsymbol{\rho}, t) = \langle B_L(\mathbf{r} + \boldsymbol{\rho}, t) B_L(\mathbf{r}, t) \rangle \quad (7)$$

is the magnetic field correlator, and

$$\begin{aligned} b(\rho) &= n_i n_j (D_{ij}(0) - D_{ij}(\rho)), \\ c(\rho) &= \frac{1}{2} n_i \Pi_{jk} F_{ijk}(\rho, \rho), \\ d(\rho) &= \left( \frac{1}{4} \Pi_{ij} \Pi_{kl} - \frac{1}{2} n_i n_k \Pi_{jl} \right) \frac{\partial}{\partial \rho_{1i}} F_{ijk}(\boldsymbol{\rho}_1, \boldsymbol{\rho}) \Big|_{\boldsymbol{\rho}_1 = \boldsymbol{\rho}}, \\ \mathbf{n} &= \boldsymbol{\rho} / \rho, \quad \Pi_{ij} = \delta_{ij} - n_i n_j, \quad B_L = \mathbf{B} \cdot \mathbf{n} \end{aligned} \quad (8)$$

represent the velocity correlators.

The applicability of the first assumption in long-term asymptotics for smooth velocity field (Batchelor regime,  $\rho < r_\nu$ ) was proved in [47]. For rough velocity field (inertial range,  $\rho > r_\nu$ ) the correctness of this assumption remains a hypothesis: we suppose that the effect of the finite correlation time is weaker than that of the asymmetry.

The insufficiency of the finite-correlation time corrections for the Kazantsev model was confirmed in DNS [48]. The second assumption produces non-physical artefacts, as any non-Gaussian model with a finite number of cumulants does [3, 50, 51]. Specifically, Eq. (6) contains the third order derivative, while the Kazantsev equation ( $c = d = 0$ ) is the second order differential equation. So, in addition to corrections to the two 'Kazantsev' modes, Eq. (6) provides

a new eigenmode that is non-physical. The account of the whole set of the higher order correlators would destroy this 'parasite' mode, but one can choose them small enough to have no significant effect on the two 'physical' modes.

For large Pm, it is enough to restrict the consideration by the viscous scales range. Eq.(6) then reduces to the equation derived in [4]. For small Pm, careful analysis of the inertial range is needed.

*Rough velocity field.* In the inertial range of hydrodynamical turbulence,  $b(\rho) \propto \rho^{4/3}$ ; indeed, for Kolmogorov turbulence [30, 32, 33] one has

$$\delta v(\rho) \propto \rho^{1/3}, \tau_c(\rho) \propto \rho/\delta v \propto \rho^{2/3} \Rightarrow b(\rho) \propto \delta v^2 \tau_c \propto \rho^{4/3}$$

Following Kazantsev [12], we hereafter consider a more general rough velocity field with the power-law structure function  $b(\rho) \propto \rho^{1+s}$ . From dimensional analysis we then get

$$\left\{ \begin{array}{l} \delta v(\rho) \propto \rho^s \\ \tau_c(\rho) \propto \rho^{1-s} \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} b(\rho) \propto \rho^{1+s} \\ c(\rho) \propto \zeta \rho^{2+s} \\ d(\rho) \propto -\beta \zeta \rho^{1+s}, \end{array} \right. \quad (9)$$

Thus, we define

$$\zeta = \frac{c(\rho)}{\rho b(\rho)}, \quad \zeta\beta = -\frac{d(\rho)}{b(\rho)} \quad (10)$$

The  $\zeta$  and  $\beta$  are dimensionless parameters that characterize the third order correlator and that are constants inside the hydrodynamic inertial range of scales;  $\zeta$  must be small to ensure the applicability of the  $V^3$  model, while  $\beta$  is, generally, not restricted. From the analysis [2] of numerical simulations [53, 54] it follows that  $\beta \simeq 2$ . We note that in (9),(10) we neglect possible intermittency: there is no evidence of anomalous scaling for time-integrated functions.

We also introduce the convenient length scale

$$r_d = \rho (2\kappa/b(s))^{1/(1+s)}, \quad \rho \gg r_\nu$$

For  $\rho \gg r_d$ , the molecular diffusion is negligibly small, and one can omit the first bracket in (6). Then

$$\begin{aligned} \frac{1}{2\kappa} \frac{\partial}{\partial t} G &= \left( \frac{\rho}{r_d} \right)^{1+s} \left( G'' + \frac{5+s}{\rho} G' + \frac{(4+s)(1+s)}{\rho^2} G \right) + \\ &+ \zeta \left( \frac{\rho}{r_d} \right)^{1+s} \left( 2\rho G''' + (18+3s)G'' + \frac{3(5+s)(2+s)-8(\beta-1)}{\rho} G' + \frac{(4+s)((4+s)(1+s)-4\beta)}{\rho^2} G \right), \quad \rho \gg r_d \end{aligned} \quad (11)$$

*Generation threshold.* In the case  $\zeta = 0$ , (6) turns into the Kazantsev equation. The magnetic field generation in this model is possible for  $s > 0$ , while for  $s \leq 0$  the magnetic field decreases [12]. Indeed, setting  $\frac{\partial}{\partial t} G = 0$  in (11) we get two stationary modes, which for  $\rho \gg r_d$  obey the scaling law:

$$G = \rho^{-\frac{s+4}{2}} (C_+ \rho^{\eta_{K+}} + C_- \rho^{\eta_{K-}}), \quad \eta_{K\pm} = \pm \frac{\sqrt{3}}{2} \sqrt{-s(s+4)} \quad (12)$$

The exponential in time solutions

$$G = e^{\Gamma t} G(\rho), \quad \partial G / \partial t = \Gamma G \quad (13)$$

of the Kazantsev equation exist if and only if the stationary solution oscillates. (This can be derived from the oscillation theorem, keeping in mind that the Kazantsev equation reduces to a Schrödinger equation; less formally, this implies that the solution (12) must match both with the 'viscous' solution at  $\rho < r_d$  and with the large-scale solution at  $\rho \rightarrow \infty$ .)

We now use the same approach to find the magnetic field generation threshold for  $\zeta \neq 0$ . Taking  $\rho \gg r_d$  for the stationary mode of (11) we get the power-law solutions  $G \propto \rho^{-\frac{s+4}{2} + \eta_{\pm}}$  with the characteristic equation for the powers  $\eta_{\pm}$ :

$$\chi(\eta) = \left( \eta^2 + \frac{3}{4}s(s+4) \right) (1 + 2\zeta\eta) - 8\zeta\beta\eta = 0 \quad (14)$$

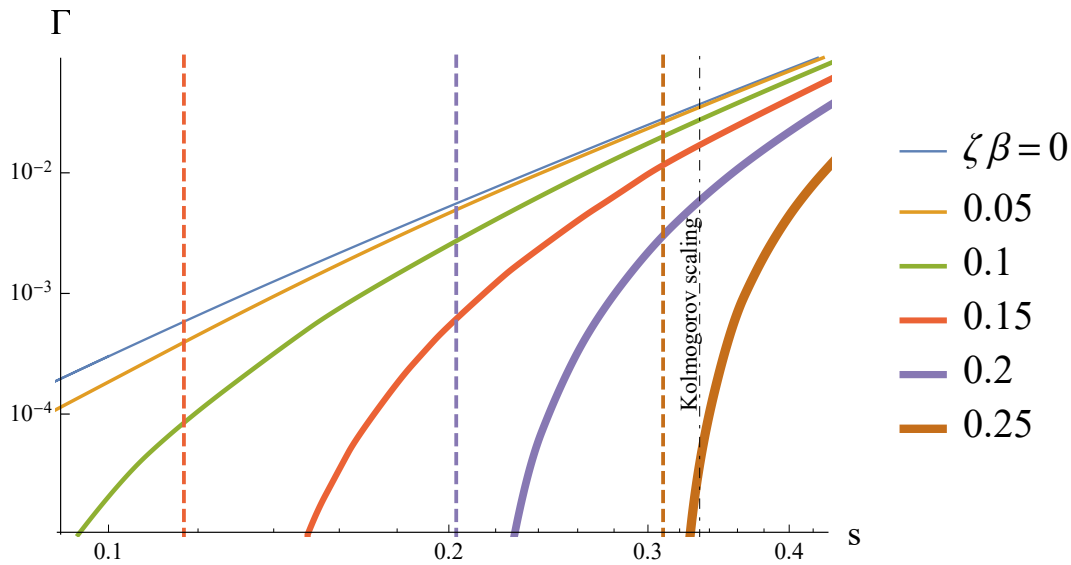


FIG. 1: The maximal growth rate  $\Gamma$  as a function of  $s$  for different values  $\zeta\beta$  in the minimal  $V^3$  model;  $\Gamma$  is normalized by  $DPm^{(1-s)/(1+s)}$  where  $D$  is characteristic frequency of small-scale eddies,  $D = -\frac{3}{4} \frac{\partial^2}{\partial \rho^2} \langle v_L(\rho)v_L(\mathbf{0}) \rangle \Big|_{\rho=0}$ .

Two of the solutions are close to those found for  $\zeta = 0$ . The third solution is  $\sim \zeta^{-1}$ , and it is a non-physical artefact. Again, the existence of generation corresponds to oscillating stationary mode eigenfunctions, i.e., to imaginary roots of (14). The minimum of the function  $\chi(\eta)$ , to the second order in  $\zeta$ , is

$$\chi_{min} = \frac{3}{4}s(s+4) - \zeta^2 \left( 4\beta - \frac{3}{4}s(s+4) \right)^2$$

We see that the presence of  $\zeta$  of any sign lowers the minimum. The generation is still possible if  $\chi_{min} > 0$ , i.e., if

$$|\zeta| < \zeta_{cr} = \frac{\sqrt{\frac{3}{4}s(s+4)}}{|4\beta - \frac{3}{4}s(s+4)|} \simeq \frac{\sqrt{3s}}{4|\beta|} \quad (15)$$

For  $s = 1/3$ ,  $\beta = 2$  we get  $\zeta_{cr} \simeq 1/7$ . For larger  $\zeta$ , there is no dynamo.

We note that the role of  $\beta$  is crucial: actually, one can see from (14) that the terms  $\zeta\beta$  are more effective to damp the oscillations than the terms with  $\zeta$  alone. This means that the influence of  $d(\rho)$  in (6) is more important than of  $c(\rho)$ .

*Growth increment* Even if the generation is not completely suppressed, its rate can be significantly slowed down. To investigate the dependence of the generation rate on the time irreversibility, we focus on the terms in (6) that contain  $d$ , and set  $c = 0$ . We call this 'minimal model'. The estimate of  $\zeta_{cr}$  obtained in the 'minimal model' coincides with (15) to the first order in  $s$ .

We are looking for exponentially growing solutions (13). Now, the 'minimal model' equation is a second-order ordinary differential equation containing the growth increment  $\Gamma$ ; it has a discrete spectrum of solutions for  $\Gamma > 0$  if  $\zeta$  satisfies (15).

By means of [55, Appendix A], we find numerically the maximal possible  $\Gamma$  for given  $s$  and  $\zeta$ , assuming  $\beta = 2$ . We normalize it by  $Pm^{\frac{1-s}{1+s}}$ , so that it becomes independent of the regularization as  $Pm \rightarrow 0$  [38]. The results are presented in Fig.1. We see that they differ significantly from the Kazantsev case: for  $s = 1/3$  the generation suppression is essential even for  $\zeta\beta \lesssim 0.2$ . The vertical asymptotes in the logarithmic coordinates correspond to the generation threshold. The dependence of the critical value of  $\zeta\beta$  on  $s$  is given in Fig.2. (For analytical expression, see Supplementary.) It confirms a good agreement between the  $V^3$  and the 'minimal'  $V^3$  models (with and without  $c(\rho)$ ) up to  $s = 1/3$ . The whole analysis is performed in the limit of infinite integral scale, i.e.,  $Rm = \infty$ ; thus, from Fig.2 it follows that, in the frame of our model, for  $\zeta\beta \gtrsim 1/3$  generation is impossible for any magnetic Reynolds number if  $Pm$  is small enough.

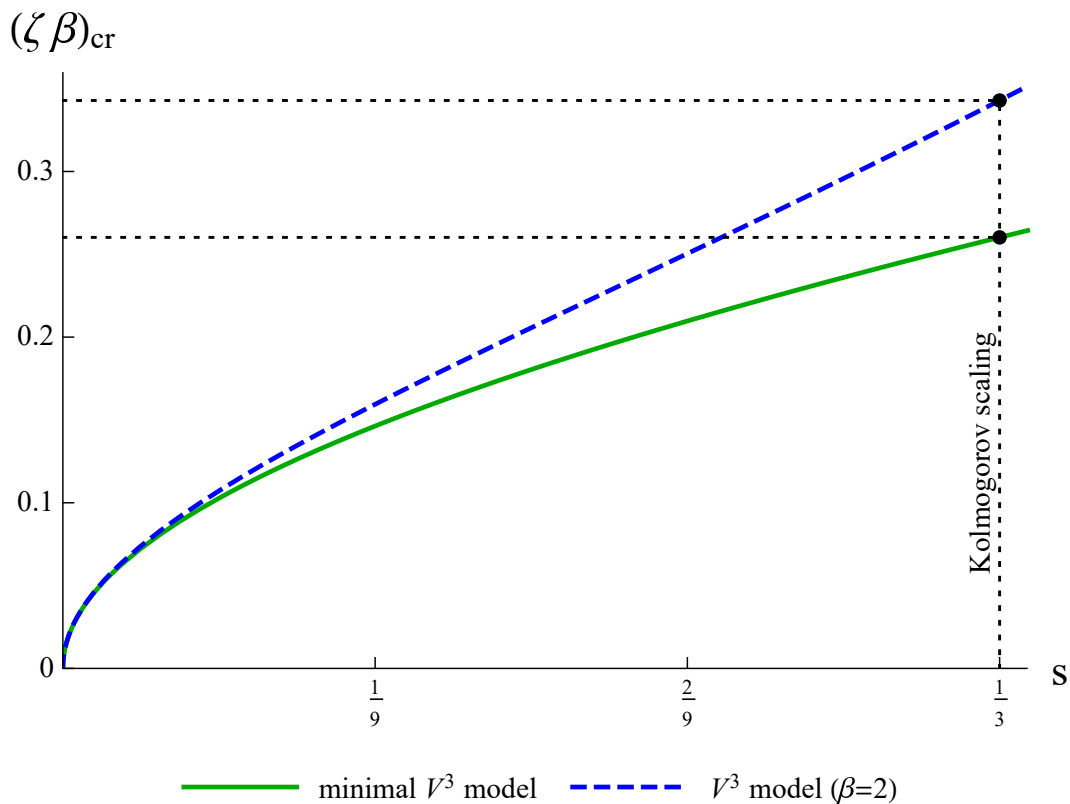


FIG. 2: The critical value  $(\zeta\beta)_{cr}(s)$  calculated in the frame of 'minimal'  $V^3$  model and  $V^3$  model (with  $\beta = 2$ ).

Summarizing, the presence of even small irreversibility is shown to suppress partially or completely the magnetic field generation. The mixed velocity and gradients correlator  $d(\rho)$  is shown to play a crucial role in this anti-dynamo effect. The effect is strong enough to be significant, e.g., for small-scale solar dynamo.

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**SUPPLEMENTARY NOTE 1. GENERALIZED KAZANTSEV EQUATION**

It is well-known that solenoidality and statistical homogeneity and isotropy in space restrict two point correlator to one scalar function:

$$\langle B_i(\mathbf{r} + \boldsymbol{\rho}, t) B_j(\mathbf{r}, t) \rangle = G(\rho, t) n_i n_j + \frac{\rho}{2} G'_\rho(\rho, t) (\delta_{ij} - n_i n_j)$$

Small-scale kinematical dynamo theory usually considers development of this correlator in time. Kazantsev show that the closed equation on  $G(\rho, t)$  can be derived when the velocity field is stationary, Gaussian and delta-correlated in time. In this case all statistical properties of the velocity field are governed by the pair correlator (see **(3)** in the main text):

$$D_{ij}(\mathbf{0}) - D_{ij}(\boldsymbol{\rho}) = b(\rho) \delta_{ij} + \frac{\rho}{2} b'(\rho) (\delta_{ij} - n_i n_j) \quad (S1)$$

To derive a dynamical equation on  $G(\rho, t)$  in the frame of 'V<sup>3</sup> model' two additional two-point correlators are needed:

$$F_{ijk}(\boldsymbol{\rho}, \boldsymbol{\rho}) = \iint \langle v_i(\mathbf{r}, t) v_j(\mathbf{r} + \boldsymbol{\rho}, t + \tau_1) v_k(\mathbf{r} + \boldsymbol{\rho}, t + \tau_2) \rangle d\tau_1 d\tau_2,$$

$$\frac{\partial}{\partial \rho_{1l}} F_{ijk}(\boldsymbol{\rho}_1, \boldsymbol{\rho}) \Big|_{\boldsymbol{\rho}_1 = \boldsymbol{\rho}} = \iint \langle v_i(\mathbf{r}, t) A_{jl}(\mathbf{r} + \boldsymbol{\rho}, t + \tau_1) v_k(\mathbf{r} + \boldsymbol{\rho}, t + \tau_2) \rangle d\tau_1 d\tau_2$$

where  $A_{jl} = \partial v_j / \partial r_l$  is a velocity gradient tensor. The first of the correlators can be restricted in our case by the only scalar function [1]:

$$F_{ijk}(\boldsymbol{\rho}, \boldsymbol{\rho}) = c(\rho) n_i \delta_{jk} - \left( c(\rho) + \frac{\rho}{2} c'(\rho) \right) (n_j \delta_{ik} + n_k \delta_{ij}) - (c(\rho) - \rho c'(\rho)) n_i n_j n_k \quad (S2)$$

The second one requires one more scalar function [2]:

$$\begin{aligned} \frac{\partial}{\partial \rho_{1l}} F_{ijk}(\boldsymbol{\rho}_1, \boldsymbol{\rho}) \Big|_{\boldsymbol{\rho}_1 = \boldsymbol{\rho}} = & -\frac{c(\rho)}{2\rho} \delta_{il} \delta_{jk} - d(\rho) n_i n_j \delta_{kl} + \left( d(\rho) - \frac{c'(\rho)}{2} \right) \delta_{ij} \delta_{kl} + \left( \frac{c(\rho)}{\rho} - d(\rho) + c'(\rho) \right) \delta_{ik} \delta_{jl} + \\ & + \left( \frac{c(\rho)}{\rho} + d(\rho) - c'(\rho) \right) n_i n_k \delta_{jl} - \left( \frac{c(\rho)}{2\rho} + 2d(\rho) - \frac{c'(\rho)}{2} \right) n_k n_l \delta_{ij} + \\ & + \left( \frac{c(\rho)}{2\rho} - \frac{c'(\rho)}{2} \right) (n_i n_l \delta_{jk} + n_j n_k \delta_{il}) - \left( \frac{3c(\rho)}{2\rho} - \frac{3c'(\rho)}{2} + \frac{\rho c''(\rho)}{2} \right) n_i n_j n_k n_l + \\ & + \left( -\frac{c(\rho)}{2\rho} + 2d(\rho) + \frac{c'(\rho)}{2} + \frac{\rho c''(\rho)}{2} \right) n_j n_l \delta_{ik} \end{aligned} \quad (S3)$$

Let us introduce the notation:

$$B_\alpha = B_\alpha(\mathbf{r}, t), \quad B'_\alpha = B_\alpha(\mathbf{r}', t), \quad v_\alpha = v_\alpha(\mathbf{r}, t), \quad v_\alpha^1 = v_\alpha(\mathbf{r}_1, t), \quad v_\alpha^2 = v_\alpha(\mathbf{r}_2, t),$$

$$\partial_\alpha = \frac{\partial}{\partial r_\alpha}, \quad \partial'_\alpha = \frac{\partial}{\partial r'_\alpha}, \quad \partial_\alpha^\rho = \frac{\partial}{\partial \rho_\alpha}, \quad \partial_\alpha^1 = \frac{\partial}{\partial r_{1\alpha}}, \quad \partial_\alpha^2 = \frac{\partial}{\partial r_{2\alpha}}$$

Then from the induction equation (1) one can obtain:

$$\frac{\partial}{\partial t} \langle B_\alpha B'_\beta \rangle = -e_{\alpha mn} e_{npq} \partial_m^\rho \langle v_p B_q B'_\beta \rangle - e_{\beta mn} e_{npq} \partial_m^\rho \langle v_p B_q B'_\alpha \rangle + 2\kappa \partial_m^\rho \partial_m^\rho \langle B_\alpha B'_\beta \rangle$$

where  $e_{npq}$  is Levi-Civita symbol. The latter equation is non-closed. However the procedure of splitting [3] can make it the closed one. A detailed performance of this procedure in the frame of V<sup>3</sup> model is given in [4, Appendix B] for smooth velocity field. The case of rough velocity field is quite analogous, so we just point out some key moments and differences between two cases. The splitting procedure is based on Furutsu-Novikov formula, which can be significantly

simplified in the frame of  $V^3$  model ( $\delta/\delta v_i$  means a functional derivative):

$$\begin{aligned}
\langle v_p B_q B'_r \rangle &= \sum_{N=1}^{\infty} \frac{1}{N!} \int d\mathbf{r}_1 dt_1 \dots d\mathbf{r}_N dt_N \langle v_p(\mathbf{r}, t) v_{i_1}(\mathbf{r}_1, t_1) \dots v_{i_N}(\mathbf{r}_N, t_N) \rangle_c \left\langle \frac{\delta^N (B_q(\mathbf{r}, t) B_r(\mathbf{r}', t))}{\delta v_{i_1}(\mathbf{r}_1, t_1) \dots \delta v_{i_N}(\mathbf{r}_N, t_N)} \right\rangle \\
&= \int d\mathbf{r}_1 dt_1 \langle v_p(\mathbf{r}, t) v_{i_1}(\mathbf{r}_1, t_1) \rangle \left\langle \frac{\delta (B_q(\mathbf{r}, t) B_r(\mathbf{r}', t))}{\delta v_{i_1}(\mathbf{r}_1, t_1)} \right\rangle \\
&\quad + \frac{1}{2} \int d\mathbf{r}_1 dt_1 d\mathbf{r}_2 dt_2 \langle v_p(\mathbf{r}, t) v_{i_1}(\mathbf{r}_1, t_1) v_{i_2}(\mathbf{r}_2, t_2) \rangle \left\langle \frac{\delta^2 (B_q(\mathbf{r}, t) B_r(\mathbf{r}', t))}{\delta v_{i_1}(\mathbf{r}_1, t_1) \delta v_{i_2}(\mathbf{r}_2, t_2)} \right\rangle \\
&= \frac{1}{2} \int d\mathbf{r}_1 D_{pj}(\mathbf{r} - \mathbf{r}_1) \left( \left\langle \frac{\delta B_q}{\delta v_j^1} B'_r \right\rangle + \left\langle B_q \frac{\delta B'_r}{\delta v_j^1} \right\rangle \right) \\
&\quad + \frac{1}{6} \int d\mathbf{r}_1 d\mathbf{r}_2 X_{pj k}(\mathbf{r} - \mathbf{r}_1, \mathbf{r} - \mathbf{r}_2) \left( \left\langle \frac{\delta^2 B_q}{\delta v_j^1 \delta v_k^2} B'_r \right\rangle + \left\langle \frac{\delta B_q}{\delta v_j^1} \frac{\delta B'_r}{\delta v_k^2} \right\rangle + \left\langle \frac{\delta B_q}{\delta v_k^2} \frac{\delta B'_r}{\delta v_j^1} \right\rangle + \left\langle B_q \frac{\delta^2 B'_r}{\delta v_j^1 \delta v_k^2} \right\rangle \right)
\end{aligned}$$

Finding the functional derivatives (see [4]) one obtain the following equation:

$$\begin{aligned}
\frac{\partial}{\partial t} \langle B_\alpha B'_\beta \rangle &= 2\kappa \partial_m^\rho \partial_m^\rho \langle B_\alpha B'_\beta \rangle - \\
&- (\epsilon_{\alpha mn} \epsilon_{npq} \delta_{\beta r} + \epsilon_{\beta mn} \epsilon_{npq} \delta_{\alpha r}) e_{j_1 j_2 k_1} \partial_m^\rho \partial_{i_1}^\rho \left( -e_{q i_1 j_1} D_{pj}(\mathbf{0}) \langle B_{k_1} B'_r \rangle + e_{r i_1 j_1} D_{pj}(\boldsymbol{\rho}) \langle B_q B'_{k_1} \rangle \right) - \\
&- (\epsilon_{\alpha mn} \epsilon_{npq} \delta_{\beta r} + \epsilon_{\beta mn} \epsilon_{npq} \delta_{\alpha r}) e_{j_1 j_2 k_1} e_{j_2 k_2 k_1} \times \\
&\quad \times \partial_m^\rho \left( 2e_{q i_1 j_1} e_{r i_2 j_2} \partial_{i_2}^\rho \left( \frac{\partial}{\partial \rho_{i_1}} F_{kpj}(\boldsymbol{\rho}_1, \boldsymbol{\rho}) \Big|_{\boldsymbol{\rho}_1 = \boldsymbol{\rho}} \langle B_{k_1} B'_{k_2} \rangle - F_{kpj}(\boldsymbol{\rho}) \partial_{i_1}^\rho \langle B_{k_1} B'_{k_2} \rangle \right) + \right. \\
&\quad \left. + e_{r i_1 j_1} e_{k_1 i_2 j_2} \partial_{i_1}^\rho \left( \frac{\partial}{\partial \rho_{i_2}} F_{pj k}(\boldsymbol{\rho}_1, \boldsymbol{\rho}) \Big|_{\boldsymbol{\rho}_1 = \boldsymbol{\rho}} \langle B_q B'_{k_2} \rangle - F_{pj k}(\boldsymbol{\rho}) \partial_{i_2}^\rho \langle B_q B'_{k_2} \rangle \right) \right).
\end{aligned}$$

Substituting (S1), (S2) and (S3) into the latter expression and convoluting enormous number of summands with the aid of computer algebra, one arrive at the generalized Kazantsev equation (6).

Note that in the case of smooth velocity field considered in [4] functions  $c$  and  $d$  are bounded by the symmetries of the flow:  $c(\rho)/d(\rho) = -\rho/6$  or  $\beta = 6$ . In the case of the rough field kinematical reasons are insufficient to obtain the relation between two functions [2]. If one suppose that the correlation times of  $\langle v_i(\mathbf{r}, t) v_j(\mathbf{r} + \boldsymbol{\rho}, t + \tau_1) v_k(\mathbf{r} + \boldsymbol{\rho}, t + \tau_2) \rangle$  and  $\langle v_i(\mathbf{r}, t) A_{jl}(\mathbf{r} + \boldsymbol{\rho}, t + \tau_1) v_k(\mathbf{r} + \boldsymbol{\rho}, t + \tau_2) \rangle$  coincide, the numerical simulation data gives  $\beta \simeq 2$ . [2]

## SUPPLEMENTARY NOTE 2. CRITICAL ANTIDYNAMO IRREVERSIBILITY IN $V^3$ MODEL

It is convenient to introduce a notation

$$\sigma = \frac{3s(s+4)}{4}.$$

Consider (14). The minimum of cubic parabola must have a non-negative value to have no generation. The following condition  $\zeta > \zeta_{cr}(s, \beta)$  can be obtained analytically. In minimal  $V^3$  model ( $\zeta\beta \gg \zeta$ ) the cubic parabola degenerates into the quadratic one and the solution is obvious:

$$(\zeta\beta)_{cr} = \frac{\sqrt{\sigma}}{4}$$

In general  $V^3$  model the analytical solution can also be found:

$$\zeta_{cr} = \frac{\sqrt{\sigma^2 + 10\beta\sigma - 2\beta^2 + 2\sqrt{\beta}(\beta + 2\sigma)^{3/2}}}{2(4\beta - \sigma)^{3/2}}$$

These dependences are depicted in Figure 2 in the main text.

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